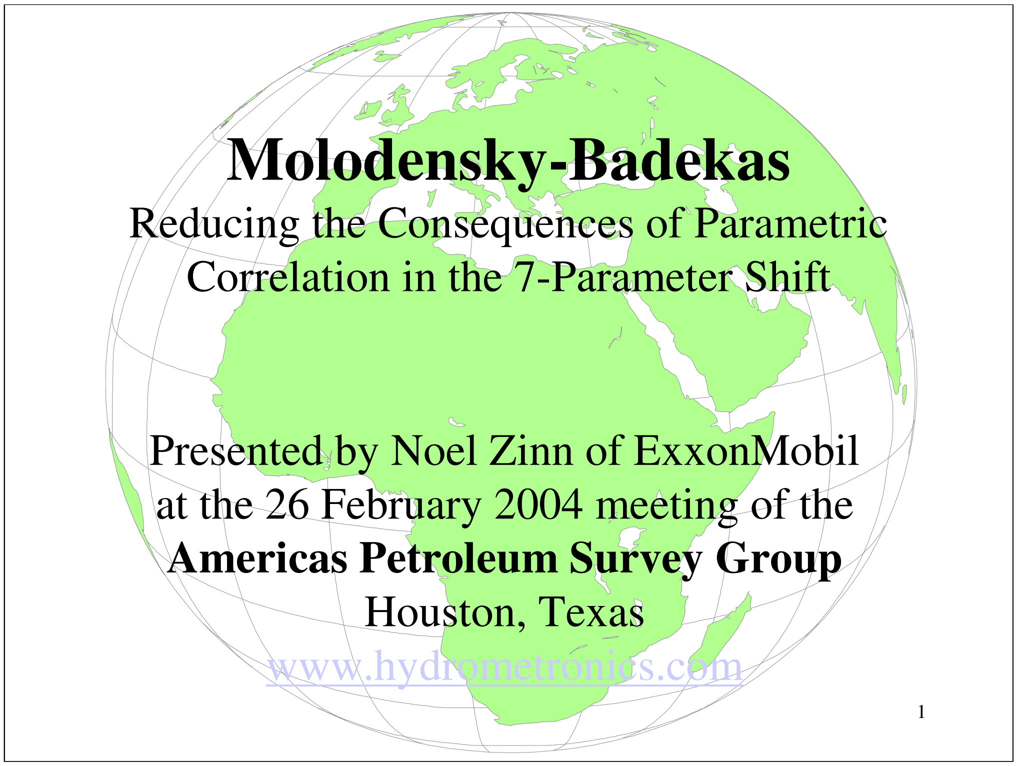


•My presentation this morning is about the advantages of deriving Molodensky-Badekas (M-B) multi-parameter datum shifts (transformations) over Helmert datum shifts in small areas where the Helmert parameters are highly correlated.

Acknowledgements

- Web site with DMA Cartesian graphics at https://www. navigator.navy.mil/navigator/ coordinate_datum_transformations.ppt
- Web site with NGA Molodensky Model http://earth-info.nima.mil/GandG/datums/wgsdt.html
- Trimble GPS Survey Planning Software http://www.trimble.com/planningsoftware.html
- Technical peer review and suggestions by Roger Lott (EPSG Geodesy Working Group) - no endorsement of author's opinions implied

2

•The Navy site has a DMA PowerPoint presentation on datum shifts with some excellent graphics that I've used.

•I've drawn an important graphic and text from the NGA (National Geospatial-Intelligence Agency) site. The NGA was formerly NIMA and before that the DMA.

•I've used a graphic produced by Trimble's GPS Survey Planning Software.

•Roger Lott of the European Petroleum Survey Group's Geodesy Working Group, and a longtime industry colleague, has been extremely helpful by reviewing this presentation for technical content, by providing useful suggestions on its presentation and by providing data for analysis. This acknowledgement does not imply Roger's endorsement of the author's opinions drawn from the technical arguments presented herein.

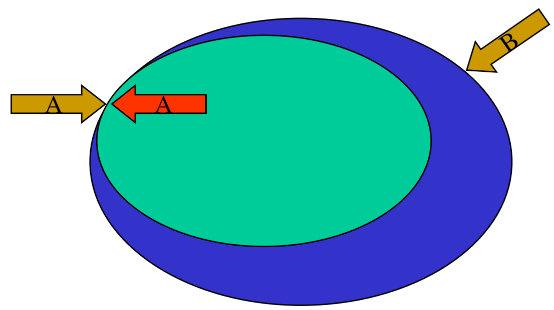
Topics

- **Statement of the issues**
- Geographicals to Cartesians to Geographicals
- Heuristic explanation of correlation problem
- The Molodensky Model at the geocenter (Helmert)
- The derivation of a 7-parameter shift
- Monte Carlo correlation w.r.t. datum area
- The Molodensky Model at the surface (M-B) solves correlation problem
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

3

•(Brief preview of the topics.)

Transformation Error



The shift required depends on the location on the datum. It is not constant.

4

•We’ve all seen something similar to this graphic from the DMA .ppt.

•Two ellipsoids are shown, presumably WGS84 and one from a local datum.

•They are related by 3 geocentric translations.

•The fit at point “A” is great, but the fit deteriorates as one moves away from “A”.

•The shift required depends on the location on the datum. It is not constant.

Issues

- 3-D 3-parameter shifts are perceived as not accurate enough, even in small datum areas.
- 2-D MRE and interpolation software (e.g., NADCON, NTv2) are not widely supported.
- 3-D 7-parameter Helmert shifts are supported.
- Do we derive 7-p Helmert shifts over small areas?
- Are these 7-p Helmert shifts really "better" than 3-p shifts in a small datum area? Or "worse"?
- Is Molodensky-Badekas a good alternative for deriving a 7-p shift in a small datum area?
- Reversibility in Molodensky-Badekas.

5

•(Read and comment.)

•(Distinguish MRE and bi-linear interpolation.)

Topics

- Statement of the issues
- **Geographicals to Cartesians to Geographicals**
- Heuristic explanation of correlation problem
- The Molodensky Model at the geocenter (Helmert)
- The derivation of a 7-parameter shift
- Monte Carlo correlation w.r.t. datum area
- The Molodensky Model at the surface (M-B) solves correlation problem
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

•(Read)

Transformation Route

Geographical (ϕ/λ/ht) on Ellipsoid1

See Appendix

Geocentric Cartesian (X/Y/Z)

Helmert or M-B Transformation

Geocentric Cartesian (X'/Y'/Z')

See Appendix

Geographical (ϕ'/λ'/ht') on Ellipsoid2

7

•7-parameter shifts - be they of the Helmert or M-B variety - operate in the Cartesian, geocentric, XYZ domain.

•Typically, however, our desired inputs and outputs are geographical coordinates (latitude, longitude and height).

•There are well-known equations for transforming geographicals into Cartesians and Cartesians into geographicals.

•You can find these equations in the Appendix.

•The rest of this talk will concentrate on the geocentric XYZ domain.

•Helmert and M-B transformations operate in XYZ.

•Helmert and M-B transformations may have fewer than 7 parameters. That is, some of the parameters may be zero.

•When only the 3 translations are involved, the Helmert transformation - used with the equations in the Appendix - are an alternative to the Standard and Abridged Molodensky equations that operate directly on geographical coordinates.

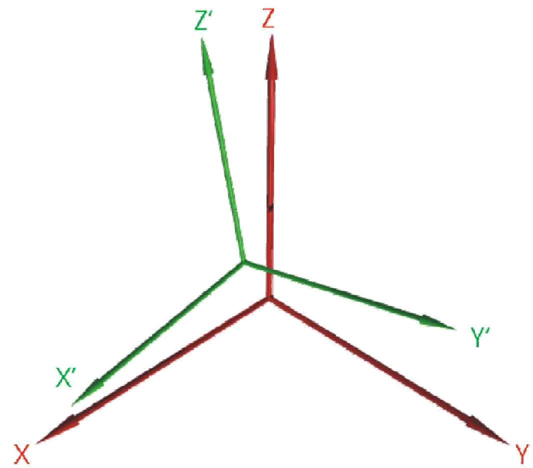
Topics

- Statement of the issues
- Geographicals to Cartesians to Geographicals
- **Heuristic explanation of correlation problem**
- The Molodensky Model at the geocenter (Helmert)
- The derivation of a 7-parameter shift
- Monte Carlo correlation w.r.t. datum area
- The Molodensky Model at the surface (M-B) solves correlation problem
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

8

•(Read)

Datum Transformation

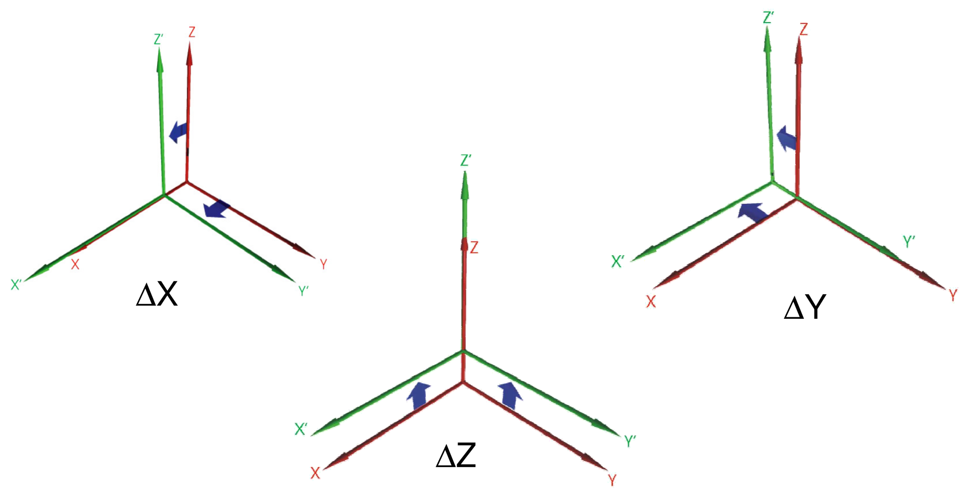


9

•In that Cartesian reference frame now, we see a compound transformation from the red frame to the green frame consisting of seven components.

Translations (3 Parameters)

Movement of points along an Axis

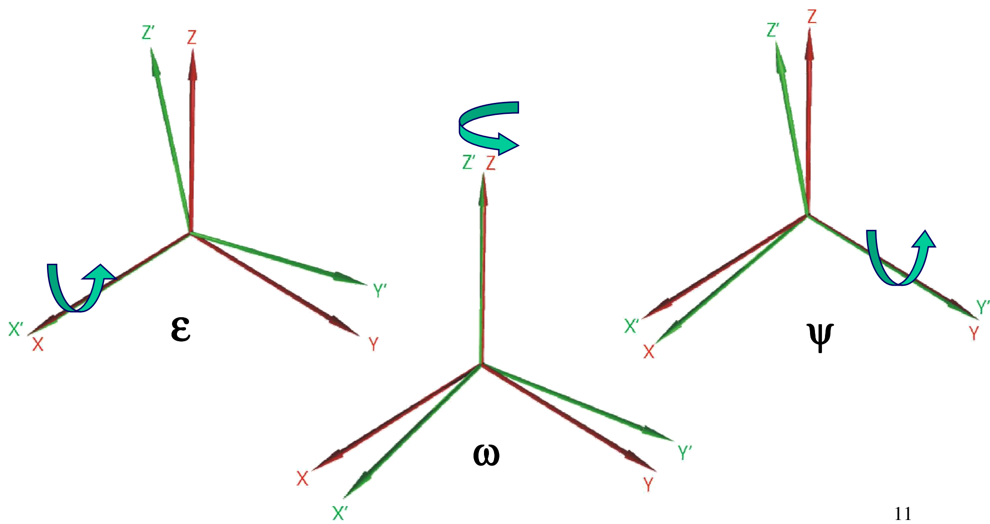


10

•Three of those components are lateral translations in each of the three axes, X, Y, and Z.

Rotations (3 Parameters)

Movement of points <u>around</u> an Axis

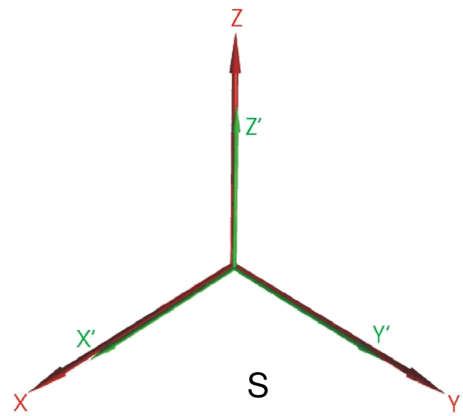


11

•Three rotations are possible, each around one of the three axes.

Scale (1 Parameter)

Changing the distance between points



12

•Finally, a change in scale is possible.

•This may open a can of worms with a few British in the group today, but here goes! Many of you will recognize this sculpture on the Prime Meridian at Greenwich. The old observatory that housed Airy's transit is to your back.

•If asked what the longitude is at the Prime Meridian, most would respond 0-0-0. It was true that the Greenwich <u>Astronomic</u> Meridian was 0-0-0. However, since the component of deflection in the prime vertical (the east-west direction) is about 5.7 seconds w.r.t. the global satellite datums, the Greenwich <u>Geodetic</u> Meridian was about 5.7 seconds West (or about 100m difference at that latitude) in the WGS72 system.

•There's more to the story. Many of you will recognize the two rotations around the Z axis in the upper left of the slide. The first was a correction to the old TRANSIT precise ephemeris (NWL9D) to bring WGS72 into alignment with Greenwich. Between WGS72 and WGS84, the BIH, the world's timekeeper located in Paris, changed the astronomic meridian of Greenwich based on an adjustment of the deflections at about 70 observatories located around the world. We all know from British nautical history that time is intimately linked to longitude, hence the need for the adjustment. The result of this adjustment at Greenwich was a 0.554 second rotation around Z require to bring WGS84 into alignment with the BIH Zero Meridian, also shown above.

•So, what is longitude of the Greenwich <u>Astronomic</u> Meridian today?

Geodetic Interpretation of the 7 Parameters w.r.t. WGS84

- **RZ** - change in longitude w.r.t. BIH Zero Meridian
- **ΔX, ΔY, ΔZ** - lateral translations at the geocenter
- **ΔS** - change in scale of linear unit w.r.t. VLBI
- **RX, RY** - your guess! Changes horizontal orientation w.r.t. CIO north. The effects of these parameters are location-specific.

14

•The Greenwich story makes the point that the 7 geocentric parameters are not mere numerical coefficients as in a Multiple Regression Equation (MRE). They are parameters imbued with geodetic significance.

•We've already discussed rotation around Z, alignment with the BIH Zero Meridian, the longitude reference for WGS84.

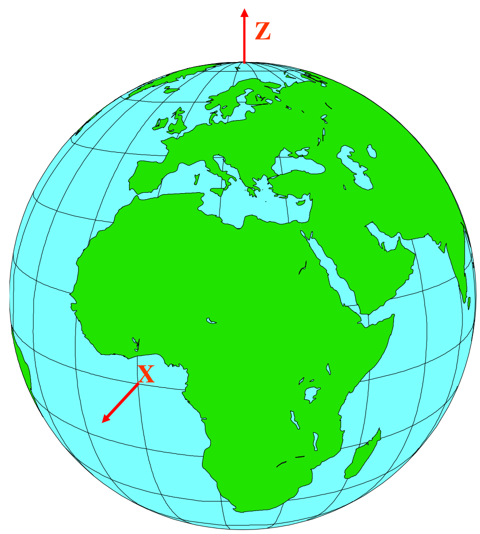
•The delta X, Y, and Z relate to the center of the earth as best our gravity models can determine it.

•WGS84 uses the same linear scale as VLBI, which is different than TRANSIT, for example, and many local datums throughout the word. Hence, the scale change parameter.

•The problem children among these 7 parameters are RX and RY. In some sense they redefine north w.r.t. the Conventional International Origin adopted by WGS84. But their effect is very location specific. These 2 parameters have - in the past - been avoided in the published transformations among the global satellite datums.

•Again, the message is that most of these parameters have geodetic interpretations. They are not "just" numerical coefficients.

X-Axis Near West Africa



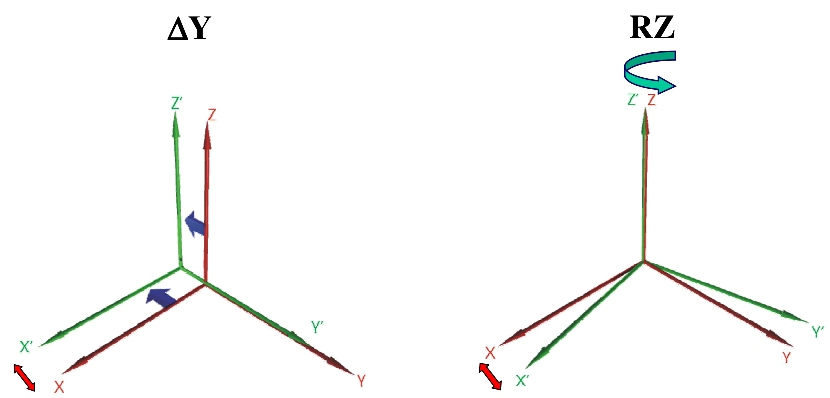
15

•Now, on to correlation among the 7 geocentric datum-shift parameters.

•For reference the globe is draped over the Cartesian reference frame.

•We’ll concentrate on that area in the Gulf of Guinea off West Africa where the X axis intersects the ellipsoid.

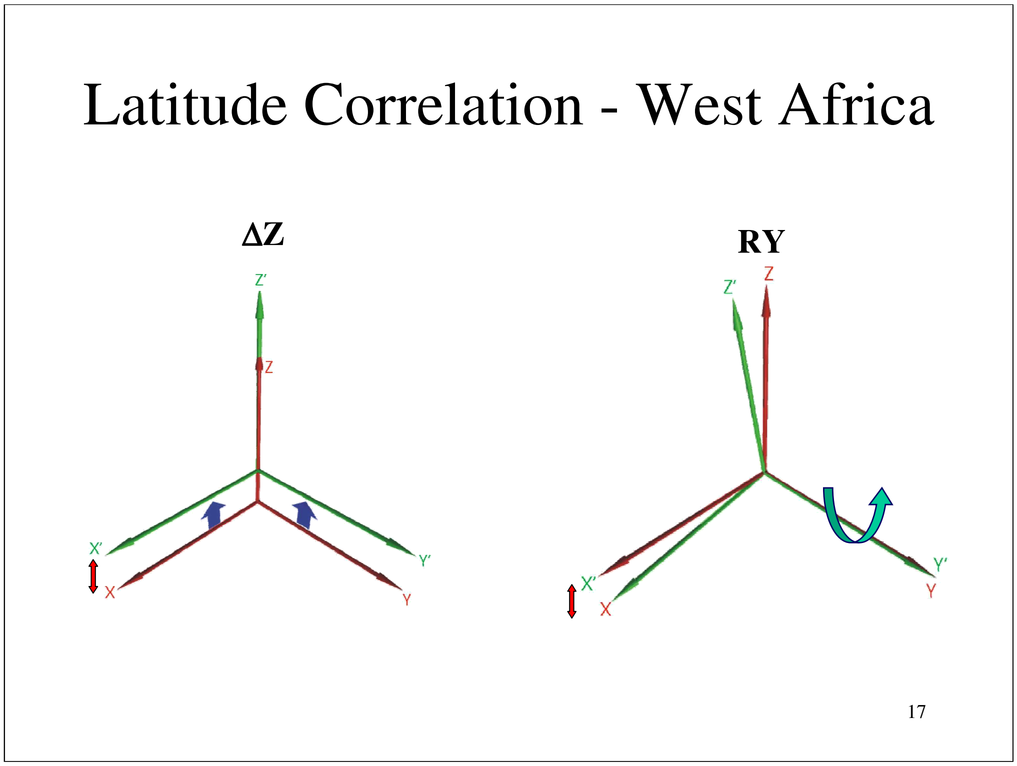
Longitude Correlation - West Africa



16

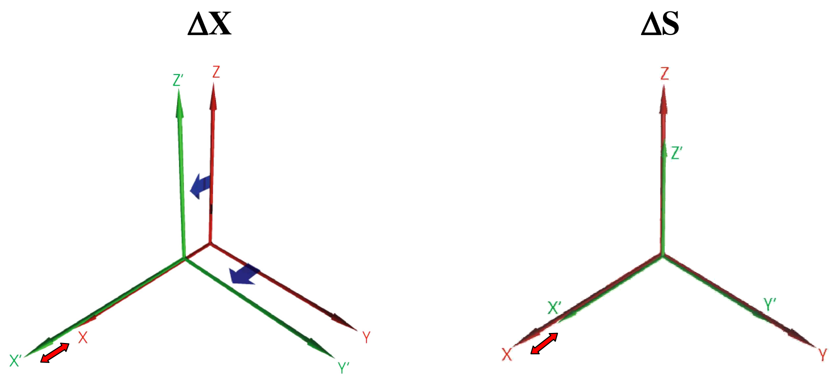
•Now we get to the heart of the problem with correlation.

•Notice that one can change longitude in the Gulf of Guinea in two ways: (1) by a translation along the X-axis, and (2) by a rotation around the Z-axis. In that local area they are equivalent.



•Next, one can change latitude in two ways: either by (1) a translation in the Z axis or by (2) a rotation about the Y axis. In that local area they are equivalent.

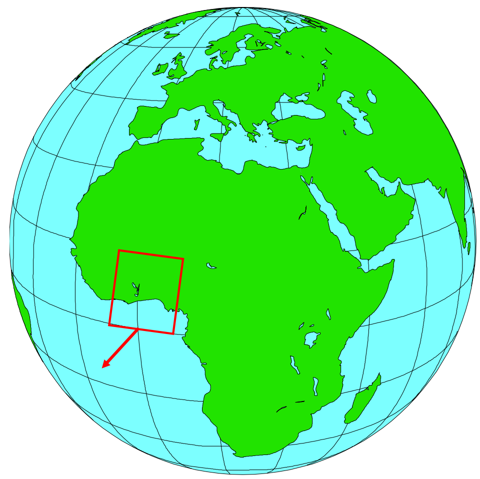
Height Correlation - West Africa



18

•Finally, one can change height in two ways: by (1) a translation along the X axis or by (2) a scale change. In that local area they are equivalent.

North Orientation - West Africa



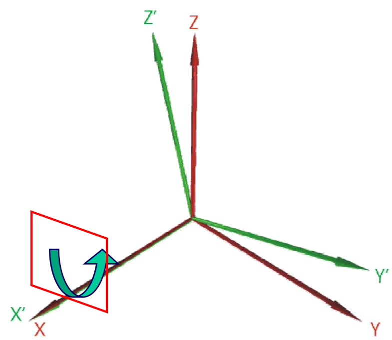
19

•That leaves one parameter, a rotation around the X axis.

•Its effect is to change the orientation of the local datum on the Gulf of Guinea at the X axis (shown as the red rectangle) w.r.t. north.

North Orientation - West Africa

RX



20

•Here we see that process in the Cartesian frame.

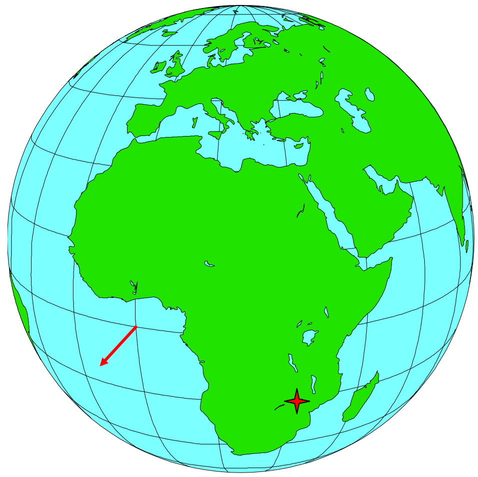
•This is a useful feature.

•For example, in the 1980s I processed TRANSIT Doppler data collected on the Portuguese concrete monuments of the Camacupa datum in Angola. I discovered that Camacupa had a scale change of 60 ppm and a north reorientation of about 5 arc seconds w.r.t. the TRANSIT broadcast ephemeris. One reason we have so many 3-p datum shifts in our lease blocks along the Angolan coast is because of this scale change and these 5 seconds of reorientation. It would be useful to represent those 5 seconds by one or more parameters in the horizontal plane.

•This can be accomplished on one of the axes (as in the Gulf of Guinea) without changing the geocentric translations.

•The Helmert rotations are geocentric, i.e., not in the horizontal plane.

Harare, Zimbabwe



21

•Now, let's move away from the X axis and the Gulf of Guinea to a point on the continent. Harare is a place where all 7 parameters interact.

Five 3-Parameter Shifts at Harare

ARC 1950 — S 28-00-00.0000, E 31-00-00.0000, 0.0000m — to — S 28-00-01.6119, E 30-59-59.8721, 24.1673m — WGS84

	DMA	#2	#3	#4	#5
ΔXm	-143	0	0	0	0
ΔYm	- 90	0	0	0	-26.540
ΔZm	-294	0	0	0	0
RX"pv	0	21.2927	-5.8558	0	0
RY"pv	0	0	9.2754	12.7939	12.5529
RZ"pv	0	-9.5105	0	-3.6077	-2.7095
ΔSppm	0	1.8965	1.8965	1.8965	0

22

•The location chosen is 28S, 31E and zero height in ARC1950.

•First, I took the DMA's published 3-p shift for that area and computed the WGS84 coordinates shown here.

•Next, I derived four other, equally-valid 3-p shifts from among the 7 available parameters. You can verify for yourselves that they all work at that point. These are position-vector rotations.

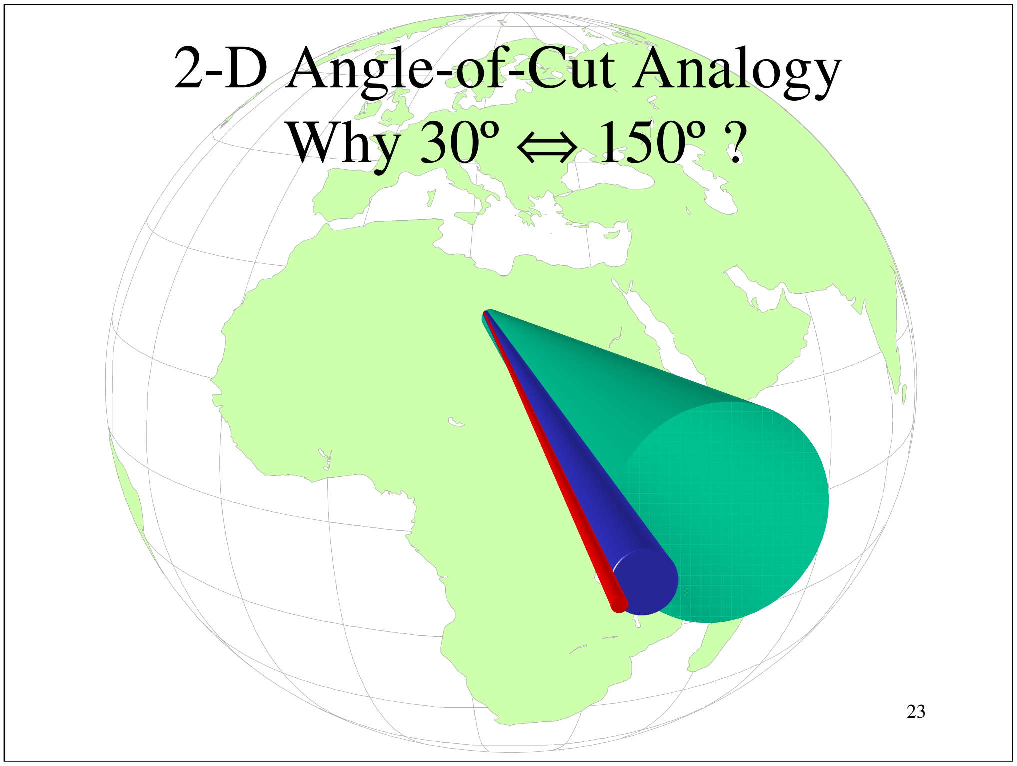
• In fact, many other 3-p shifts are possible.

•Notice that if two rotations are used - and it doesn't matter which two - then the scale change parameter is obligatory to modify height. Rotations work only on the surface of the ellipsoid. Height changes are off the surface.

•But notice that if we include a translation - and it doesn't matter which one - then we don't need the scale parameter.

•Why is this possible? It's possible because the problem is over-parameterized. At a point, very different parameters have the same effect. They are correlated at Harare just as they are correlated in the Gulf of Guinea.

•The questions we'll address in much of the rest of this presentation are: (1) How large of an area does it take to avoid adverse effects of this correlation? and, (2) Is there another way?



•The gray beards among us will remember the angle of cut guideline back in radio navigation days. Why did we require an angle of cut between 30 and 150 degrees? Because the closer you get to 180 or 0 degrees, such as crossing a baseline or its extension, the problem becomes increasingly indeterminate.

•Technically, it's a judgement about acceptable HDOP, the geometrical multiplier of range error in position error expressed as dRMS. HDOP is related to the trace of the covariance matrix. We'll talk more about covariance and correlation matrices in this presentation.

•Which of the areas represented by the cones radiating from the geocenter is adequate for the job of successfully deriving a 7-p Helmert datum shift?

•Haven't we decided that the red cone - which is analogous to the point in Harare - is inadequate? There are too many possibilities, too much correlation. Is the blue cone adequate? Or the green cone? How much surface area in the local datum do we really need to derive a good 7-p solution?

•We'll have some answers by the end of this presentation.

Topics

- Statement of the issues
- Geographicals to Cartesians to Geographicals
- Heuristic explanation of correlation problem
- **The Molodensky Model at the geocenter (Helmert)**
- The derivation of a 7-parameter shift
- Monte Carlo correlation w.r.t. datum area
- The Molodensky Model at the surface (M-B) solves correlation problem
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

24

•(Read)

NGA Molodensky Model - 1

http://earth-info.nima.mil/GandG/datums/wgsdt.html

Seven-Parameter geometric transformation

MOLODENSKY Model

The transformation is between a non-global local datum and a geocentric global geodetic system. The rotations are to be considered about the three axes at the "initial" point of the local datum. The scale factor is also considered with respect to the initial point.

25

- These words are taken directly from the NGA web site cited.
- (Read)

NGA Molodensky Model - 2

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i = \begin{bmatrix} U \\ V \\ W \end{bmatrix} + \begin{bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{bmatrix} + \begin{bmatrix} 0 & \omega & -\psi \\ -\omega & 0 & \varepsilon \\ \psi & -\varepsilon & 0 \end{bmatrix} \begin{bmatrix} U - U' \\ V - V' \\ W - W' \end{bmatrix}_i + \Delta S \begin{bmatrix} U - U' \\ V - V' \\ W - W' \end{bmatrix}_i$$

where "i" denotes any point common to the local datum and geodetic system and the (U',V',W') are the coordinates of the "initial" point of the local datum. The three angles correspond to the small rotations taken positive in the counter clock-wise mode, when viewed from the end of the respective axes (at the "initial" point) towards the origin.

Note: When the "initial" point of the local datum (U',V',W') is not provided, assume values of (0,0,0).

26

•(Continue reading)

•When the initial point is at geocenter, the matrix equation given here reduces to the Helmert model. Notice that the NGA uses "coordinate-frame" rotations.

•When the initial point is somewhere other than the geocenter, the equation here is commonly called the Molodensky-Badekas (M-B).

•The literature advises that the M-B initial (or evaluation) point be the barycenter of the Cartesian coordinates of the points in the local datum. That is not obligatory. It can be the fundamental point of the local datum, or it can be anywhere nearby.

•The barycenter is the mathematical average of the Cartesian coordinates of all the points in the data set.

•The barycenter has the advantage of reducing most correlations to zero.

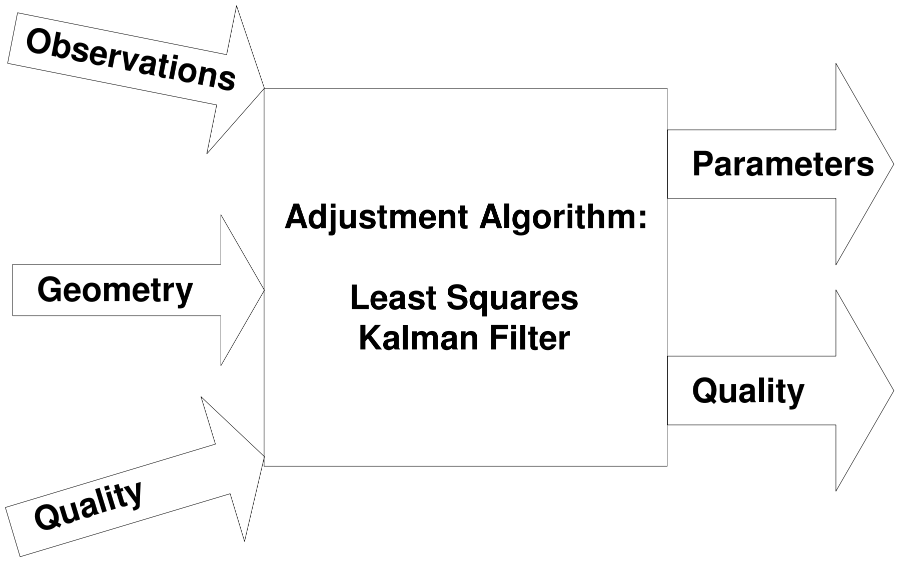
Topics

- Statement of the issues
- Geographicals to Cartesians to Geographicals
- Heuristic explanation of correlation problem
- The Molodensky Model at the geocenter (Helmert)
- **The derivation of a 7-parameter shift**
- Monte Carlo correlation w.r.t. datum area
- The Molodensky Model at the surface (M-B) solves correlation problem
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

27

•We've seen the adjustment model. Now, how do we derive a 7-p shift?

Adjustment Inputs and Outputs



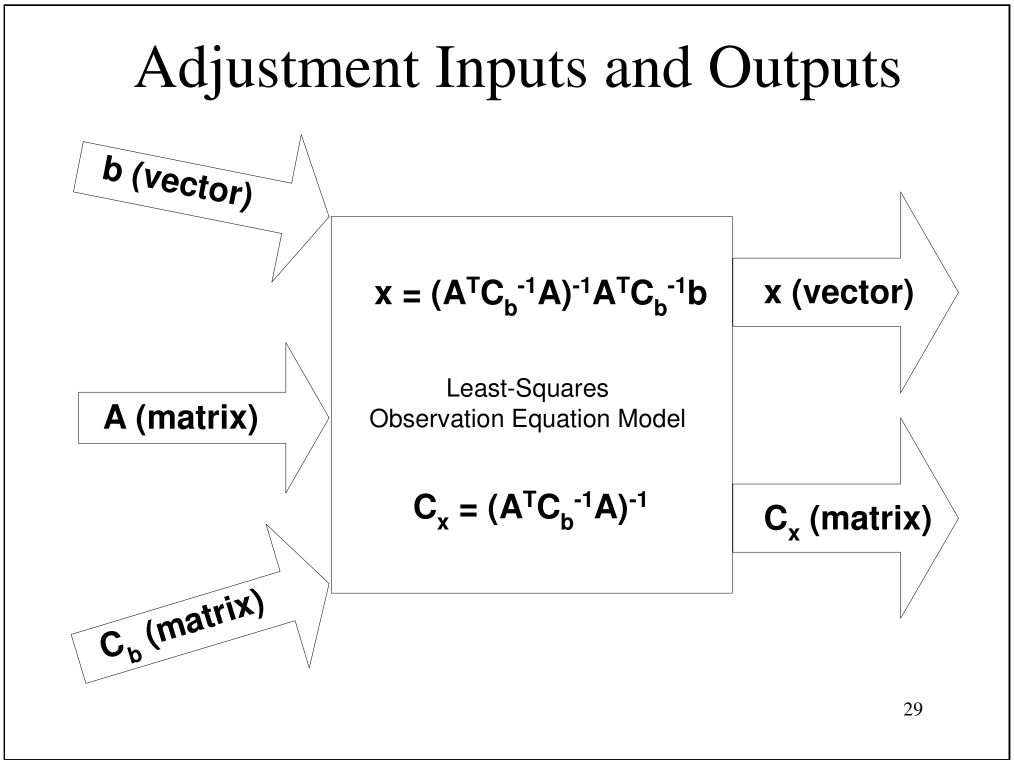
28

•The derivation of a 7-p shift in the XYZ domain is an adjustment of the coordinates of a common set of survey points in two datums that solves for the values of the 7 parameters that relate them best in a least-squares sense.

•The observation equations (which we've already seen in the Molodensky Model) are linear, so no iteration is required.

•Like any adjustment our inputs are our observations (which are the coordinate differences), our geometry (which is the spatial relationship of the points in the design matrix), and the quality of the observation (the "accuracy" of our survey points).

•Our outputs are the parameters themselves and the quality of the parameters represented by the covariance matrix, which we'll transform into a correlation matrix.



•Here are the actual linear-algebra equations that do the job.

•The coordinate differences are contained in the “b” vector.

•The “A” matrix contains the design or geometry of the adjustment, the “angle of cut”, so to speak.

•“Cb” is the coordinate quality matrix.

•After least-squares processing as shown, the parameters are contained in “x”,

•And the quality of those parameters in “Cx”.

Snippet of Derivation Code

```
% Populate the design matrix (A) and the observations vector (b)
for this = 1:pts
   A(this*3-2, :) = [1  0  0  0          -Zfrom(this) Yfrom(this) Xfrom(this)];
   A(this*3-1, :) = [0  1  0  Zfrom(this) 0          -Xfrom(this) Yfrom(this)];
   A(this*3,   :) = [0  0  1 -Yfrom(this) Xfrom(this) 0           Zfrom(this)];

   b(this*3-2) =  Xto(this) - Xfrom(this);
   b(this*3-1) =  Yto(this) - Yfrom(this);
   b(this*3)   =  Zto(this) - Zfrom(this);
end

% Define covariance matrix of the observations.  Assume 1-m SD error per axis.
Cb = eye(pts*3);

% Compute covariance matrix of the parameters.
Cx = inv(A'*inv(Cb)*A);

% Solve for the 7 parameters.  Rotations are in radians.
x = Cx*A'*b

% Solve for post-adjustment residuals.
b2 = A*x-b;

% Solve for variance factor
vf = b2'*b2/(pts*3-length(x));

% Scale Cx by vf, solve for SDs of parameters and the correlation matrix
SD = (diag(Cx*vf)).^.5;
correlation = cor(Cx)
```

30

•This is a snippet of code run in Matlab or O-Matrix to do the job. I offer it because, in these matrix-manipulation languages, the equations map almost directly into code, which you can see for yourselves.

•Notice the 7 columns that go into the “A” matrix. One can solve for fewer parameters by knocking out one or more columns, as I did in the Harare case to solve for only 3 parameters. Solving for fewer parameters (6 or 5) lowers correlation, which is a good thing.

•The quality matrix (Cb) is an identity matrix (eye), implying 1 meter of random error in each axis.

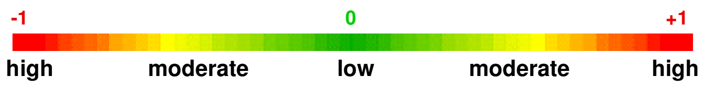
•Notice that, using the post-adjustment residuals, the code solves for the variance factor (vf), used to scale the covariance matrix (Cx), and then solves for the standard deviations.

•Finally, the code turns the covariance matrix into a correlation matrix with a function call to “cor” (noted in blue).

Brief Review of Correlation

Normalizing the Covariance Matrix (Cx)

$$\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$



31

•Because the benefits of the M-B over the Helmert for small-area derivations have to do with correlations, I offer a brief review of what correlation is mathematically.

•The correlation coefficient “rho” is simply the off-diagonal covariance “sigma_ij” divided by the connected standard deviations on the diagonal, “sigma_i” and “sigma_j”.

•Correlation “rho” varies between -1 and +1. Numbers near zero indicate low correlation, which is good. Numbers near -1 and +1 indicate high correlation, which is bad.

•High correlation is diagnostic of “over parameterization” in an adjustment, similar to the situation I described for the Gulf of Guinea, where latitude, longitude and height changes could each be made in two different ways.

•Because of over parameterization at a point in Harare, I was able to solve for multiple 3-p shifts from among the 7 available parameters.

•High correlations are related to near singularity in matrix inversion, a “divide by zero” problem.

•In Kalman filter terminology, this is referred to as “poor observability”.

•In least-squares adjustments, this is an “ill-conditioned” problem.

Topics

- Statement of the issues
- Geographicals to Cartesians to Geographicals
- Heuristic explanation of correlation problem
- The Molodensky Model at the geocenter (Helmert)
- The derivation of a 7-parameter shift
- **Monte Carlo correlation w.r.t. datum area**
- The Molodensky Model at the surface (M-B) solves correlation problem
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

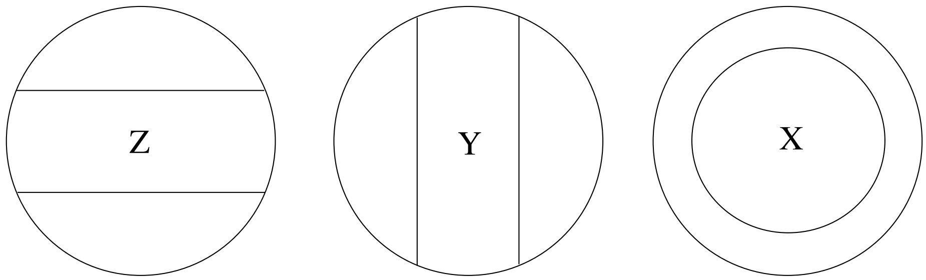
32

•Now we're going to look at some correlation matrices and other numbers generated over differently-sized datum areas with both normally and uniformly distributed random numbers.

•This is called "Monte Carlo" analysis.

Distribution of 1M Random Points

Ellipsoid Thirds Along Each Axis



332688	333336	333758
335854	333009	332650
331458	333655	333592

33

•First I checked my random number generation technique by populating the world with one million uniformly-scattered survey points.

•Then I divided the world into thirds three different ways, along each of the three Cartesian axes.

•It's interesting to note that if you divide a sphere's diameter into thirds, the surface area of each of the two "caps" equals the area of the mid belt.

•If my technique were perfect (and, thus, not really random!) you would see mostly "3s" except for three "4s" at the ends.

•Instead, you see a pretty good random distribution.

•Notice, however, that along the Z axis there is a slight "bulge" along the mid belt - the ellipsoid effect.

Global and Continental Areas
Induced Datum Shift and *a priori* Errors

- Δ**Xm**
- Δ**Ym**
- Δ**Zm**
- **RX”**
- **RY”**
- **RZ”**
- Δ**Sppm**

- **+700m**
- **-500m**
- **+200m**
- **-3”**
- **+5”**
- **-2”**
- **+3ppm**

- **X ±1m 1σ**
- **Y ±1m 1σ**
- **Z ±1m 1σ**

34

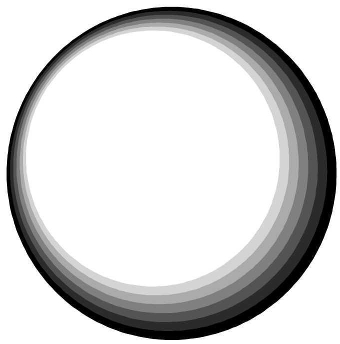
•Next, we need a datum shift to solve for in the adjustment process.

•So, I induced this shift while generating the survey points, which are uniformly distributed over the areas to be shown. I chose a “worst case” 7-p shift. Few - if any - datum shifts with so many large numbers are likely, though some may have individual parameters as large as these.

•After the (deterministic) datum shift I add 1-m, 1-sigma, normally-distributed random error on each survey point.

Entire World: Monte Carlo

19 random points, 1-m SD / axis, 7 parameters as defined



RMS = 0.98, vf = 1.09, sduw = 1.04

	7P	SD	SDsc
ΔX	699.674	0.233	0.243
ΔY	-500.062	0.232	0.242
ΔZ	199.817	0.233	0.243
RX	-2.992	0.010	0.010
RY	5.006	0.009	0.009
RZ	-1.997	0.009	0.010
ΔS	3.031	0.036	0.038

Correlation_Matrix

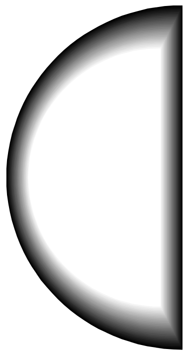
```
[ 1.00  0.00  0.00 -0.02  0.09  0.17  0.02]
[ 0.00  1.00  0.01 -0.08  0.01  0.03 -0.13]
[ 0.00  0.01  1.00 -0.17 -0.01  0.02  0.06]
[-0.02 -0.08 -0.17  1.00 -0.05 -0.11  0.00]
[ 0.09  0.01 -0.01 -0.05  1.00  0.13 -0.00]
[ 0.17  0.03  0.02 -0.11  0.13  1.00 -0.00]
[ 0.02 -0.13  0.06 -0.00  0.00  0.00  1.00]
```

35

- Now, the results!
- For this simulation I populated the world with 19 uniformly-distributed points in the first datum, induced the “worst case” datum shift just described, and applied 1 meter of normally distributed error in each axis to the result in the second datum.
- Then I solved for the 7 parameters.
- Notice that the results under “7P” are close to the induced datum shift. That’s good. The unscaled standard deviations of the 7 parameters are shown in the column under “SD”. The scaled standard deviations are shown in the column under “SDsc”. Both are similar and comparable to the differences between solved-for and induced datum shift. That’s good, too.
- Just above the parameters I show the RMS of the residuals, the variance factor (vf) and the standard deviation of unit weight (sduw), all about 1.
- Finally, in the correlation matrix below, notice that, except for the main diagonal, which is always 1, the correlations are low. This indicates good geometry.
- The message is that just 19 points uniformly distributed throughout the entire world gives a good derivation of 7 geocentric Helmert parameters.

Hemisphere: Monte Carlo

19 random points, 1-m SD / axis, 7 parameters as defined



RMS = 0.84, vf = 0.80, sduw = 0.90

	7P	SD	SDsc
ΔX	699.647	0.272	0.244
ΔY	-499.741	0.294	0.263
ΔZ	199.523	0.302	0.271
RX	-3.020	0.009	0.008
RY	5.000	0.012	0.011
RZ	-2.007	0.011	0.010
ΔS	3.020	0.042	0.038

Correlation_Matrix

```
[ 1.00 -0.01  0.07  0.00  0.21  0.02 -0.49]
[-0.01  1.00 -0.05 -0.19 -0.07 -0.61  0.00]
[ 0.07 -0.05  1.00  0.02  0.64  0.08  0.13]
[ 0.00 -0.19  0.02  1.00  0.02  0.08 -0.00]
[ 0.21 -0.07  0.64  0.02  1.00  0.12 -0.00]
[ 0.02 -0.61  0.08  0.08  0.12  1.00  0.00]
[-0.49  0.00  0.13 -0.00 -0.00 -0.00  1.00]
```

36

•Our second Monte Carlo simulation is for half the world, a hemisphere.

•Notice that the 7 parameters themselves are still close to the induced shift. The SDs have crept up only slightly.

•But now in the correlation matrix we've begun to see some real correlations, which I've marked in orange in the upper diagonal. These are caused by the translations pairing with rotations in their effect on coordinates and the hemispherical geometry not completely distinguishing the two.

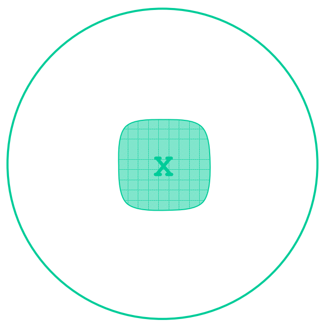
•This geometry is analogous to the one-sided geometry of GPS - no SVs below the earth - and to poor VDOP and to vertical coordinates that are poorer than the horizontal coordinates.

•We should expect the numbers across the top (RMS, vf, sduw) to be about 1. The differences with 1 seen here are simply a consequence of the small number of points in this simulation, both in their random error and in their distribution. Another simulation could be greater than 1. The greater the number of points, the more likely the standard deviation of unit weight will be close to 1.

•Still, a good result.

“Australia-Sized”: Monte Carlo

19 random points, 1m SD / axis, 7 parameters as defined



RMS = 0.85, vf = 0.83, sduw = 0.91

	7P	SD	SDsc
ΔX	698.919	1.202	1.095
ΔY	-501.506	1.696	1.544
ΔZ	201.359	1.714	1.561
RX	-2.977	0.039	0.035
RY	5.052	0.056	0.051
RZ	-1.946	0.055	0.050
Δ	3.149	0.188	0.172

Correlation_Matrix

```
[ 1.00  0.01 -0.02 -0.00 -0.04 -0.01 -0.98]
[ 0.01  1.00 -0.17  0.01 -0.17 -0.99  0.00]
[-0.02 -0.17  1.00  0.03  0.99  0.17 -0.02]
[-0.00  0.01  0.03  1.00  0.03  0.01  0.00]
[-0.04 -0.17  0.99  0.03  1.00  0.17  0.00]
[-0.01 -0.99  0.17  0.01  0.17  1.00 -0.00]
[-0.98  0.00 -0.02 -0.00 -0.00 -0.00  1.00]
```

37

•For this simulation I chose a much smaller surface area, about the size of Australia, but centered in the Gulf of Guinea.

•First notice that the SDs for the translations have crept up to over a meter and the translation parameters are off by more than meter as well. This is good news. It means that our stochastic model is competently predicting actual results.

•One of the references cited at the end of the presentation, the excellent paper by Bruce Harvey, offers guidance on the statistical testing of these parameters. Hypothesis testing is not addressed (rigorously) in this presentation.

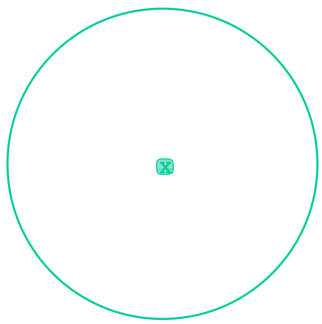
•Impressionistically, however, you may agree that there’s not much point in claiming a precision for these translations to anything better than a meter due to the metric size of the SDs. Nevertheless, we’ve identified our “worst case” datum shift pretty well.

•Look now at the three numbers in the correlation matrix highlighted in orange. In addition to the size of the area , another reason that they’re high is that this simulated area is centered around the X axis. If the area were between axes, the correlations would be distributed among the other parameters.

•The size of these correlations is related to the creeping size of the SDs as the area gets smaller.

“Cyprus-Sized”: Monte Carlo

19 random points, 1m SD / axis, 7 parameters as defined



RMS = 0.97, vf = 1.08, sduw = 1.04

	7P	SD	SDsc
ΔX	761.639	31.764	33.019
ΔY	-561.148	49.999	51.974
ΔZ	96.197	61.947	64.394
RX	-3.374	1.027	1.068
RY	1.631	2.003	2.082
RZ	-0.027	1.617	1.681
ΔS	-6.650	4.980	5.177

Correlation_Matrix

```
[ 1.00 -0.00 -0.00 -0.00 -0.00 -0.00 -1.00]
[-0.00  1.00  0.58  0.00  0.58 -1.00  0.00]
[-0.00  0.58  1.00  0.00  1.00 -0.58 -0.00]
[-0.00  0.00  0.00  1.00  0.00 -0.00  0.00]
[-0.00  0.58  1.00  0.00  1.00 -0.58  0.00]
[-0.00 -1.00 -0.58 -0.00 -0.58  1.00 -0.00]
[-1.00  0.00 -0.00 -0.00  0.00  0.00  1.00]
```

38

•Australia is big country and a small continent. Our derivation there was reasonably successful at the meter level or so with 19 points of metric quality.

•Now we push this process to the extreme with an area the size of the island of Cyprus, also centered in the Gulf of Guinea.

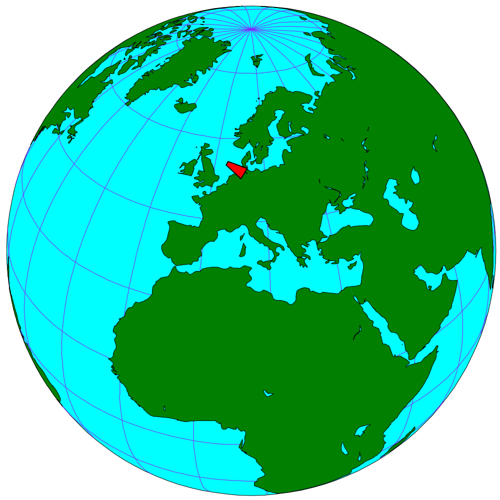
•Notice now that the SDs for the translations are in the tens of meters and the translations themselves are off just bad. Again, this correspondence of stochastic model and parametric results is good news.

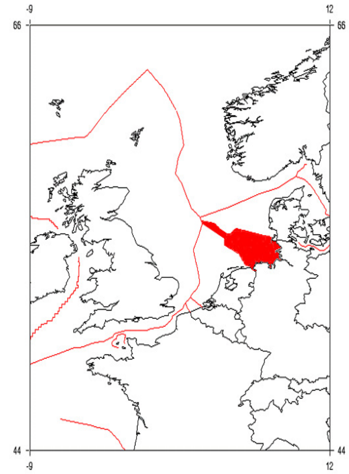
•The correlations in red are rounded to two decimal places, so they are close to, but probably not exactly, 1. These high correlations are between dX and scale, dY and RZ, and dZ and RY - exactly as we saw heuristically with the graphics in the Gulf of Guinea.

•The correlations in blue (0.58) are just the effects of the 1m of random noise on the survey points in this extremely small area. Those numbers jump around from simulation to simulation, but the correlations in red are fixed at 1.

•The smaller the area, the greater the effect of survey-point quality on parametric results. Survey point quality is magnified by poor geometry.

German North Sea





39

•Now we turn our attention to another small area, the German sector of the North Sea, pictured here, which is rich in oil reserves.

•The German North Sea is a 2.5 times the size of Cyprus.

•It is not located on a Cartesian axis, but between them.

Proposed German No. Sea 7-P Shift

Geodetic Transformations (Single-Step)

EPSG geodesy parameters

Search Criteria: **Germany - offshore North Sea.**

#2 *Transformation Code:* *Name:* **ED50 to WGS 84**

Source CRS: code = **4230** *name =* **ED50**

Source Ellipsoid:: **International 1924**

Semi-major axis (a) = **6378388**

Semi-minor axis (b) = *inverse flattening =* **297**

Target CRS: code = **4326** *name =* **WGS 84**

Target Ellipsoid:: **WGS 84**

Semi-major axis (a) = **6378137 metre**

Semi-minor axis (b) = *inverse flattening =* **298.257223563**

Data Source: **EPSG** *Change ID:* **Rev. Date:** **31-Dec-03**

Information Source:

Area of Use: **Germany - offshore North Sea.**

Scope: **Recommended transformation for Germany North Sea petroleum purposes.**

Remarks: **Approximation to better than 0.5m of transformation adopted in June 2003.**

40

•My interest in small-area 7-p derivations was re-ignited by a recent action of the EPSG Geodesy Working Group, the computation of a 7-p Helmert transformation for the German sector of the North Sea.

•This is a second transformation method proposed for the German North Sea sector. The first is the MRE equations authorized by a German agency.

•The EPSG Geodesy Working Group has derived the 7 geocentric Helmert parameters that best match the MRE results. The intent of the EPSG is to provide a better match to the MREs with 7 parameters than the 3 translations alone provide.

•The EPSG have succeeded. The 3 translations alone fit the MREs in the 1 to 2 meter range. The 7 parameters together fit the MREs at about half a meter.

•Software for 3 and 7 parameter Helmert transformations is widely available; software for the MREs is not.

•Consequently, the 7-p Helmert transformation is seen as a “practical” improvement by the EPSG.

•Nevertheless, I have technical reservations about the consequences of this proposal and the precedent it sets for others who may wish to do the same.

Proposed German No. Sea 7-P Shift

Trf Variant: ***Version:*** **EPSG-Ger Nsea**

Transformation method: **Position Vector 7-param. transformation**

For Polynomial transformation methods only:

Unit Source Offsets: ***Unit Target Offsets:***

Transformation Parameter Name	***Value***	***Unit of Measure***
X-axis translation	***-157.89***	***metre***
Y-axis translation	***-17.16***	***metre***
Z-axis translation	***-78.41***	***metre***
X-axis rotation	***2.118***	***arc-second***
Y-axis rotation	***2.697***	***arc-second***
Z-axis rotation	***-1.434***	***arc-second***
Scale difference	***-5.38***	***parts per million***

Area of Use: Germany - offshore North Sea.

Scope: Recommended transformation for Germany North Sea petroleum purposes.

Remarks: Approximation to better than 0.5m of transformation adopted in June 2003 .

41

•Here are the 7 parameters offered by the EPSG, which are highlighted in orange.

•Notice the precision of the numbers.

German North Sea

Induced Datum Shift and *a priori* Errors

- Δ**Xm** • **0m**
- Δ**Ym** • **0m**
- Δ**Zm** • **0m**
- **RX”** • **0”**
- **RY”** • **0”**
- **RZ”** • **0”**
- Δ**Sppm** • **0ppm**

- **X ±1m 1σ**
- **Y ±1m 1σ**
- **Z ±1m 1σ**

42

•Using the tools developed for this presentation, we’ll now analyze the quality of any 7-p Helmert shift derived for the German North Sea.

•At this juncture we make a important change. Rather than inducing the “worst case” shift that we’ve been using heretofore, or even the EPSG shift itself, I induce no shift at all.

•Our expectation now is to simulate data as before, solve for the 7 parameters, and find all zeros!

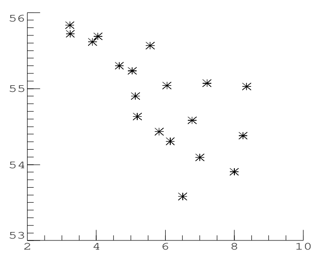
•The random positional errors are the same as before.

German North Sea: Monte Carlo 1

19 random points, 1-m SD / axis, all 7 parameters = 0

Helmert with 7 Parameters

RMS = 0.79, vf = 0.71, sduw = 0.84



	7P	SD	SDsc
ΔX	-19.379	25.595	21.549
ΔY	10.495	19.159	16.131
ΔZ	-0.541	19.138	16.113
RX	0.155	0.537	0.452
RY	0.474	0.959	0.807
RZ	-0.425	0.501	0.422
ΔS	1.705	1.884	1.586

Correlation_Matrix

[1.00	-0.65	-0.69	-0.57	-0.96	0.55	-0.27]
[-0.65	1.00	0.56	0.91	0.67	-0.77	-0.04]
[-0.69	0.56	1.00	0.47	0.86	-0.47	-0.51]
[-0.57	0.91	0.47	1.00	0.58	-0.44	0.00]
[-0.96	0.67	0.86	0.58	1.00	-0.55	0.00]
[0.55	-0.77	-0.47	-0.44	-0.55	1.00	-0.00]
[-0.27	-0.04	-0.51	-0.00	-0.00	0.00	1.00]

43

•Here is our first result.

•I used 19 points as before. These are not the same MRE-match 19 points used by the EPSG, but they are chosen randomly within constraints in the same approximate area.

•Notice first that our 7 parameters are not zeros or even really close.

•Notice the size of the SDs, somewhat smaller than Cyprus, but still uncomfortably large. Again, the SDs are competently predicting our failure to identify the induced datum shift. Why is the adjustment failing?

•The reason is, of course, high correlation among the parameters caused by the small area of our data set.

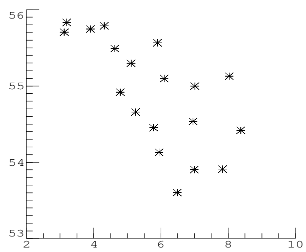
•I've highlighted correlations more than 0.6. Notice how widely distributed they are over the matrix now that we are between the axes.

•Given the small size of the German North Sea relative to the world, our 7-p adjustment is "ill conditioned" and the results are poor. It is a simple mater of geometry. We are still too close to the situation in Harare with too many correlated parameters. The problem can be (almost) solved in too many ways. We're "too close to the baseline", so to speak.

German North Sea: Monte Carlo 2

19 random points, 1-m SD / axis, all 7 parameters = 0

Helmert with 7 Parameters



RMS = 0.97, vf = 1.07, sduw = 1.03

	7P	SD	SDsc
ΔX	18.504	23.623	24.388
ΔY	-34.929	19.208	19.830
ΔZ	-11.071	17.874	18.453
RX	-0.955	0.539	0.557
RY	-0.693	0.879	0.907
RZ	0.638	0.497	0.513
ΔS	0.106	1.862	1.922

Correlation_Matrix

[1.00	-0.63	-0.65	-0.56	-0.96	0.54	-0.29]
[-0.63	1.00	0.54	0.91	0.66	-0.77	-0.04]
[-0.65	0.54	1.00	0.45	0.84	-0.44	-0.54]
[-0.56	0.91	0.45	1.00	0.58	-0.44	-0.00]
[-0.96	0.66	0.84	0.58	1.00	-0.54	-0.00]
[0.54	-0.77	-0.44	-0.44	-0.54	1.00	0.00]
[-0.29	-0.04	-0.54	-0.00	-0.00	0.00	1.00]

44

•Here is a second simulation. Slightly different points and slightly different errors in a random way.

•Notice how different the parameters are.

•Notice the high correlations.

•When an adjustment is ill conditioned, small changes in the data set can produce very different results, which we're seeing here.

•Consider the precision of these parameters.

Topics

- Statement of the issues
- Geographicals to Cartesians to Geographicals
- Heuristic explanation of correlation problem
- The Molodensky Model at the geocenter (Helmert)
- The derivation of a 7-parameter shift
- Monte Carlo correlation w.r.t. datum area
- **The Molodensky Model at the surface (M-B) solves correlation problem**
- Dilution of Precision (P7DOP)
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

45

•What do we do about this correlation problem in a small area?

•The answer is Molodensky-Badekas.

NGA Molodensky Model - 2

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i = \begin{bmatrix} U \\ V \\ W \end{bmatrix} + \begin{bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{bmatrix} + \begin{bmatrix} 0 & \omega & -\psi \\ -\omega & 0 & \varepsilon \\ \psi & -\varepsilon & 0 \end{bmatrix} \begin{bmatrix} U - U' \\ V - V' \\ W - W' \end{bmatrix}_i + \Delta S \begin{bmatrix} U - U' \\ V - V' \\ W - W' \end{bmatrix}_i$$

where "i" denotes any point common to the local datum and geodetic system and the (U',V',W') are the coordinates of the "initial" point of the local datum. The three angles correspond to the small rotations taken positive in the counter clock-wise mode, when viewed from the end of the respective axes (at the "initial" point) towards the origin.

Note: When the "initial" point of the local datum (U',V',W') is not provided, assume values of (0,0,0).

46

•This is a repeat of an earlier slide from the NGA. Read the note at the bottom.

•When the initial point (U’, V’, W’) is (0, 0. 0) the Molodensky Model is the Helmert model that we are used, and have been analyzing heretofore.

•When the initial point (U’, V’, W’) is somewhere else, the Molodensky Model is what we are calling the Molodensky-Badekas.

•The literature advises that the M-B initial (or evaluation) point be the barycenter of the Cartesian coordinates of the points in the local datum. That is not obligatory. It can be the fundamental point of the local datum, or it can be anywhere nearby.

•An initial point at the barycenter has the advantage of reducing most correlations to zero.

German North Sea
Molodensky-Badekas at Barycenter
Induced Datum Shift and *a priori* Errors

• **ΔXm**	• **0m**
• **ΔYm**	• **0m**
• **ΔZm**	• **0m**
• **RX”**	• **0”**
• **RY”**	• **0”**
• **RZ”**	• **0”**
• **ΔSppm**	• **0ppm**

- **X ±1m 1σ**
- **Y ±1m 1σ**
- **Z ±1m 1σ**

47

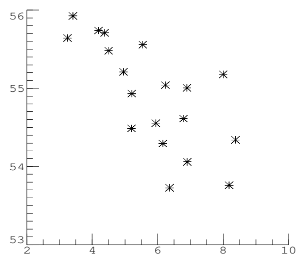
•Again, I have not induced any datum shift into our Monte Carlo data.

•Our solved parameters in this local area of the German North Sea should be all zeros.

German North Sea: Monte Carlo 3

19 random points, 1-m SD / axis, all 7 parameters = 0

Barycenter Molodensky-Badekas with 7 Parameters




```
Xc=3654621.110, Yc=373231.167, Zc=5195307.741

RMS = 0.81, vf = 0.74, sduw = 0.86
```

	7P	SD	SDsc
ΔX	-0.130	0.229	0.198
ΔY	0.070	0.229	0.198
ΔZ	-0.025	0.229	0.198
RX	-0.794	0.570	0.491
RY	-1.747	0.953	0.821
RZ	0.674	0.525	0.452
ΔS	-1.106	1.914	1.649

```
               Correlation_Matrix
[ 1.00  0.00  0.00  0.00  0.00 -0.00 -0.00]
[ 0.00  1.00  0.00  0.00  0.00 -0.00  0.00]
[ 0.00  0.00  1.00  0.00  0.00 -0.00  0.00]
[ 0.00  0.00  0.00  1.00  0.61 -0.47  0.00]
[ 0.00  0.00  0.00  0.61  1.00 -0.57  0.00]
[-0.00 -0.00 -0.00 -0.47 -0.57  1.00  0.00]
[-0.00  0.00 -0.00 -0.00 -0.00  0.00  1.00]
```

48

•Here are the results.

•Again, I used 19 points as before, chosen randomly within constraints. These are not the same 19 points used by the EPSG, or any points that used before, but they are in the same approximate area.

•In addition to the same results as before you see the Cartesian coordinates of the barycenter just below the blue subtitle.

•Notice that the translations are, indeed, close to zero.

•Notice also that the SDs for the translations are also small, in fact, two orders of magnitude smaller than they were for the Helmert model. For the translations this is a much better result.

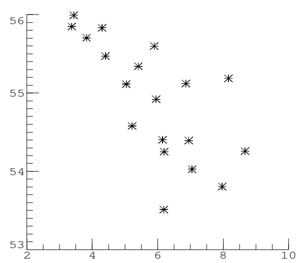
•Notice now that the SDs for the rotations and scale are still large. Although we are not addressing the issue of statistical testing in this presentation, these SDs are an appraisal of the sensitivity of M-B for the German No. Sea, of how large the rotations and scale have to be to correctly assess whether they exist at all. Just because we get numbers for the rotation and scale parameters, doesn't mean that they mean anything! In this case a proper statistical test might reject the hypothesis that rotations and scale exist at all.

•Notice the correlation matrix. All zeros now except those connecting the rotations, which I've highlighted in orange.

German North Sea: Monte Carlo 4

19 random points, 1-m SD / axis, all 7 parameters = 0

Barycenter Molodensky-Badekas with 7 Parameters



Xc=3653872.647, Yc=375185.984, Zc=5195534.393

RMS = 0.96, vf = 1.06, sduw = 1.03

	7P	SD	SDsc
ΔX	−0.199	0.229	0.236
ΔY	−0.091	0.229	0.236
ΔZ	−0.188	0.229	0.236
RX	0.163	0.546	0.562
RY	0.455	0.847	0.873
RZ	0.062	0.498	0.513
ΔS	3.679	1.814	1.868

Correlation_Matrix

```
[ 1.00 -0.00 -0.00 -0.00 -0.00  0.00 -0.00]
[-0.00  1.00  0.00  0.00  0.00 -0.00 -0.00]
[-0.00  0.00  1.00  0.00  0.00 -0.00 -0.00]
[-0.00  0.00  0.00  1.00  0.60 -0.48 -0.00]
[-0.00  0.00  0.00  0.60  1.00 -0.56 -0.00]
[ 0.00 -0.00 -0.00 -0.48 -0.56  1.00  0.00]
[-0.00 -0.00 -0.00 -0.00 -0.00  0.00  1.00]
```

49

•Here’s another simulation with slightly different points. Pretty much the same result within a couple sigma.

•Regarding the sensitivity of this technique, remember my story about the Camacupa datum in Angola about a scale change of 60ppm and north reorientation of 5 seconds w.r.t. to the broadcast ephemeris. This M-B technique is plenty sensitive to detect those changes confidently. Additionally, Angola is a larger area, meaning better geometry, meaning smaller SDs.

Proposed German No. Sea 7-P Shift

Trf Variant: ***Version:*** **EPSG-Ger Nsea**

Transformation method: **Position Vector 7-param. transformation**

For Polynomial transformation methods only:

Unit Source Offsets: ***Unit Target Offsets:***

Transformation Parameter Name	***Value***	***Unit of Measure***
X-axis translation	***-157.89***	***metre***
Y-axis translation	***-17.16***	***metre***
Z-axis translation	***-78.41***	***metre***
X-axis rotation	***2.118***	***arc-second***
Y-axis rotation	***2.697***	***arc-second***
Z-axis rotation	***-1.434***	***arc-second***
Scale difference	***-5.38***	***parts per million***

Area of Use: Germany - offshore North Sea.

Scope: ***Recommended transformation for Germany North Sea petroleum purposes.***

Remarks: ***Approximation to better than 0.5m of transformation adopted in June 2003.***

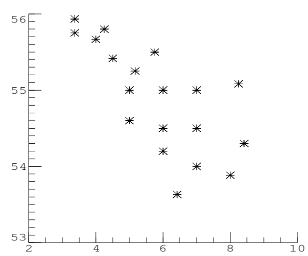
50

•Roger Lott, the former chair of the EPSG Geodesy Working Group, generously shared with me the actual 19-point data set he used to derive the parameters above, highlighted in orange.

German North Sea: MRE Match

19 MRE points, no induced random error

Helmert with 7 Parameters



RMS = 0.118, vf = 0.016, sduw = 0.126

	7P	SD	SDsc
ΔX	-157.893	24.889	3.145
ΔY	-17.164	19.882	2.512
ΔZ	-78.412	18.614	2.352
RX	2.118	0.554	0.070
RY	2.697	0.929	0.117
RZ	-1.434	0.511	0.065
ΔS	-5.380	1.871	0.236

Correlation_Matrix

```
[ 1.00 -0.66 -0.67 -0.59 -0.96  0.57 -0.27]
[-0.66  1.00  0.57  0.91  0.68 -0.78 -0.04]
[-0.67  0.57  1.00  0.48  0.85 -0.48 -0.52]
[-0.59  0.91  0.48  1.00  0.61 -0.47  0.00]
[-0.96  0.68  0.85  0.61  1.00 -0.57  0.00]
[ 0.57 -0.78 -0.48 -0.47 -0.57  1.00 -0.00]
[-0.27 -0.04 -0.52  0.00  0.00 -0.00  1.00]
```

51

•I ran those 19 EPSG points through the same software as before with no induced datum shift (obviously) and no induced random error. These parametric results agree exactly with the EPSG. Notice that the RMS, vf and sduw numbers are low. This may be impressive to some.

•Both the Helmert and M-B 7-p derivation techniques are intended to relate real survey points with real errors (both biases and random) in one datum to those same points surveyed in a different datum with (probably) different methods and, no doubt, different errors. In this EPSG case, however, we are not adjusting the survey points themselves, but a mathematical abstraction. That abstraction is the MREs for the German North Sea, which were derived from the original survey points and which mask the errors of the original survey points.

•Aside: original (geo-stationary) StarFix "pseudo ARGO" output anecdote.

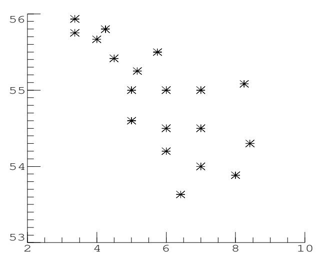
•It might be argued that the MREs are "authoritative" since they are sanctioned by a German agency, and therefore correct by definition or by agreement. But that is an axiomatic argument, not a technical argument. We shouldn't confuse the two and we shouldn't promote questionable geodesy for expediency. It sets a bad precedent for others less careful than the EPSG.

•Finally, notice that correlations are high and notice that neither the unscaled or scaled SDs support the numerical precision reported by EPSG - by orders of magnitude.

German North Sea: MRE Match

19 MRE points, no induced random error

Barycenter Molodensky-Badekas with 7 Parameters




```
Xc=3655727.054, Yc=373465.142, Zc=5194453.8

RMS = 0.118, vf = 0.016, sduw = 0.126
```

	7P	SD	SDsc
ΔX	-107.055	0.229	0.029
ΔY	-97.928	0.229	0.029
ΔZ	-150.318	0.229	0.029
RX	2.118	0.554	0.070
RY	2.697	0.929	0.117
RZ	-1.434	0.511	0.065
ΔS	-5.380	1.871	0.236

```
            Correlation_Matrix
[ 1.00 -0.00 -0.00 -0.00 -0.00  0.00 -0.00]
[-0.00  1.00  0.00  0.00  0.00 -0.00 -0.00]
[-0.00  0.00  1.00  0.00  0.00 -0.00 -0.00]
[-0.00  0.00  0.00  1.00  0.61 -0.47 -0.00]
[-0.00  0.00  0.00  0.61  1.00 -0.57 -0.00]
[ 0.00 -0.00 -0.00 -0.47 -0.57  1.00  0.00]
[-0.00 -0.00 -0.00  0.00  0.00 -0.00  1.00]
```

52

•Finally, you see the 7 M-B parameters with the same EPSG data set.

•Notice that the translations are different and that both the unscaled and scaled SDs for the translations are much lower.

•Notice that the rotations and scale and their SDs are the same. In the M-B case, however, these rotations and scale have a very different geodetic interpretation. They are applied at the barycenter, not the geocenter.

•Notice that the rotations are correlated, as highlighted in orange.

•The next step in this analysis (which I am not reporting) for either the Helmert or M-B model would be to test the significance of the rotation and scale parameters statistically and reject any that fail. RY would be a good candidate. We could then solve for a 6-p M-B. Eliminating one rotation would lower the correlations.

•That process could lead to a 3-p shift for German North Sea being the only mathematically-tenable shift given the geometry of the area and the quality of the data. Just a thought and a topic for another presentation.

•Really, the best approach is to work with the original survey data and eliminate the intermediate mathematical abstraction of the MREs, which hide the errors.

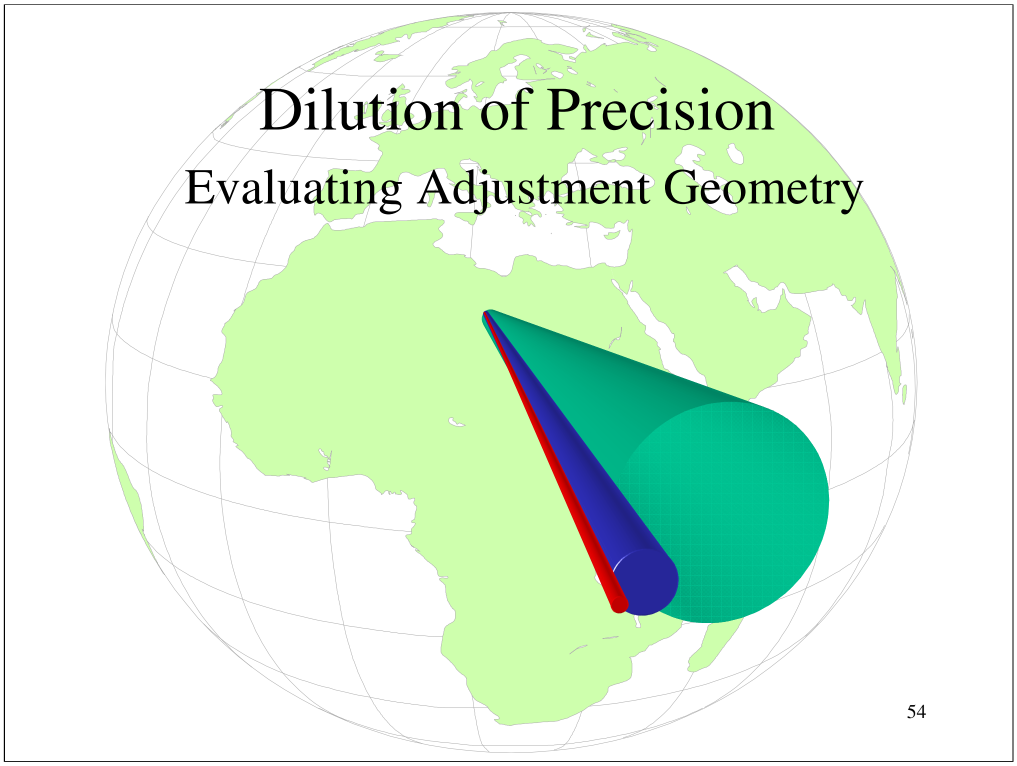
•If, on the other hand, the MREs are authoritative, then we should use the MREs.

Topics

- Statement of the issues
- Geographicals to Cartesians to Geographicals
- Heuristic explanation of correlation problem
- The Molodensky Model at the geocenter (Helmert)
- The derivation of a 7-parameter shift
- Monte Carlo correlation w.r.t. datum area
- The Molodensky Model at the surface (M-B) solves correlation problem
- **Dilution of Precision (P7DOP)**
- Conclusion
- Reversibility in Molodensky-Badekas (Appendix)

53

•Our final topic is a quantification of the geometrical component of the Helmert 7-p formulation.



•Earlier I asked how much surface area on the ellipsoid was required to provide good 7-p Helmert geometry.

•What is the Helmert analogue to HDOP in horizontal positioning and GDOP in GPS?

•That analogue is P7DOP.

Helmert 7-P Dilution of Precision

From the world of hydrography and GPS

$$HDOP = \sqrt{\sigma_E^2 + \sigma_N^2} \Big/ \sigma_0 \qquad PDOP = \sqrt{\sigma_E^2 + \sigma_N^2 + \sigma_U^2} \Big/ \sigma_0$$

$$GDOP = \sqrt{\sigma_E^2 + \sigma_N^2 + \sigma_U^2 + c^2 \cdot \sigma_T^2} \Big/ \sigma_0$$

From the world of 7-p Helmert transformations

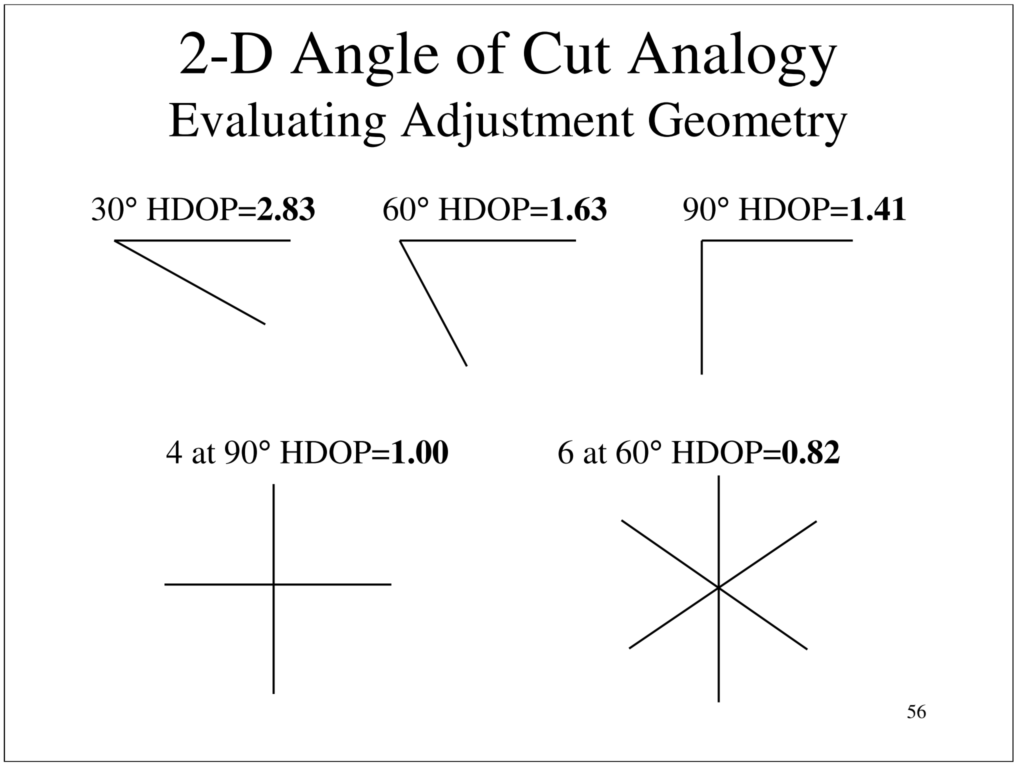
$$\boxed{P7DOP = \sqrt{\sigma_{\Delta X}^2 + \sigma_{\Delta Y}^2 + \sigma_{\Delta Z}^2 + a \cdot b \cdot (\sigma_{RX}^2 + \sigma_{RY}^2 + \sigma_{RZ}^2 + \sigma_{\Delta S}^2)} \Big/ \sigma_0}$$

55

•The concept of HDOP ("Horizontal Dilution of Precision") has been used in radio-navigation for decades. It is the square root of the trace of the covariance matrix divided by a reference standard deviation. Range quality (1m, 5m, 10m, whatever) times HDOP yields position error expressed as dRMS. HDOP captures the geometrical component of position error, i.e., "angle of cut".

•With GPS, the concept was expanded to 3 dimensions (east, north and up) to give PDOP ("Position DOP") and and to 4 dimension (adding time, i.e., receiver clock uncertainty) to give GDOP ("Geometrical DOP"). Notice that in GDOP the variance of time is multiplied by the speed of light squared for consistency of units and to relate clock uncertainty to position uncertainty.

•I applied that approach and now define P7DOP in the red box. It is a measure of the geometrical component of parameter uncertainty in the position domain. Notice that I multiply the rotation and scale variances (in radians and ppm respectively) by product of the ellipsoidal semi-axes to "map" these uncertainties to the surface. This is similar to the use of the speed of light in GDOP. This is only an approximation since the rotations and scale map differently at different locations in the world. But it's a good approximation, and used relatively to compare the geometric component of differently-sized area, it's excellent.

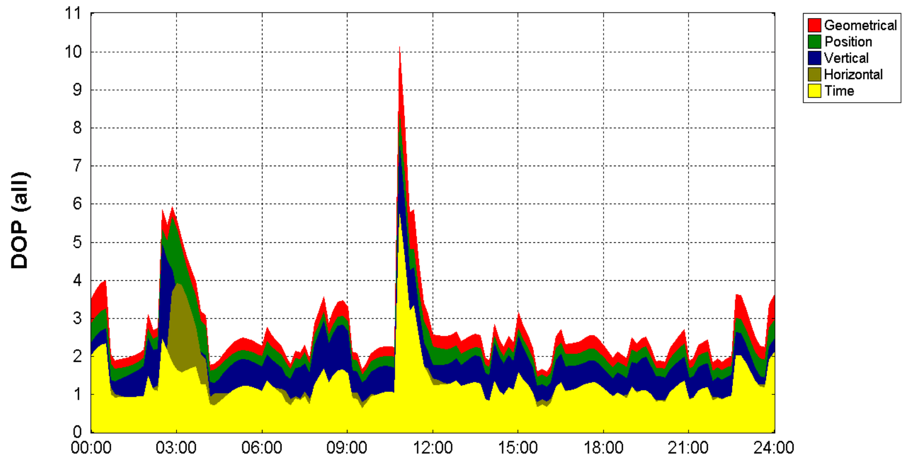


•To get a heuristic “feel” for these quantities here are several radio-navigation range configurations in the horizontal plane and their associated HDOPs.

•Notice that HDOP decreases as the angle of cut gets better and as additional ranges are added.

•Range error times HDOP gives dRMS.

All GPS DOPs: Houston 2/14/04



57

•This GPS planning software shows H, P and GDOP in the Houston area two weeks ago.

•My company requires a PDOP of less than 6 for seismic acquisition. Notice the spike where that is exceeded.

•The variation in the DOPs is due to the changing geometry of the GPS satellites and their setting and rising, i.e., the number of SVs in view.

7-Parameter Helmert

P7DOP by Area, Angle and Number of Points

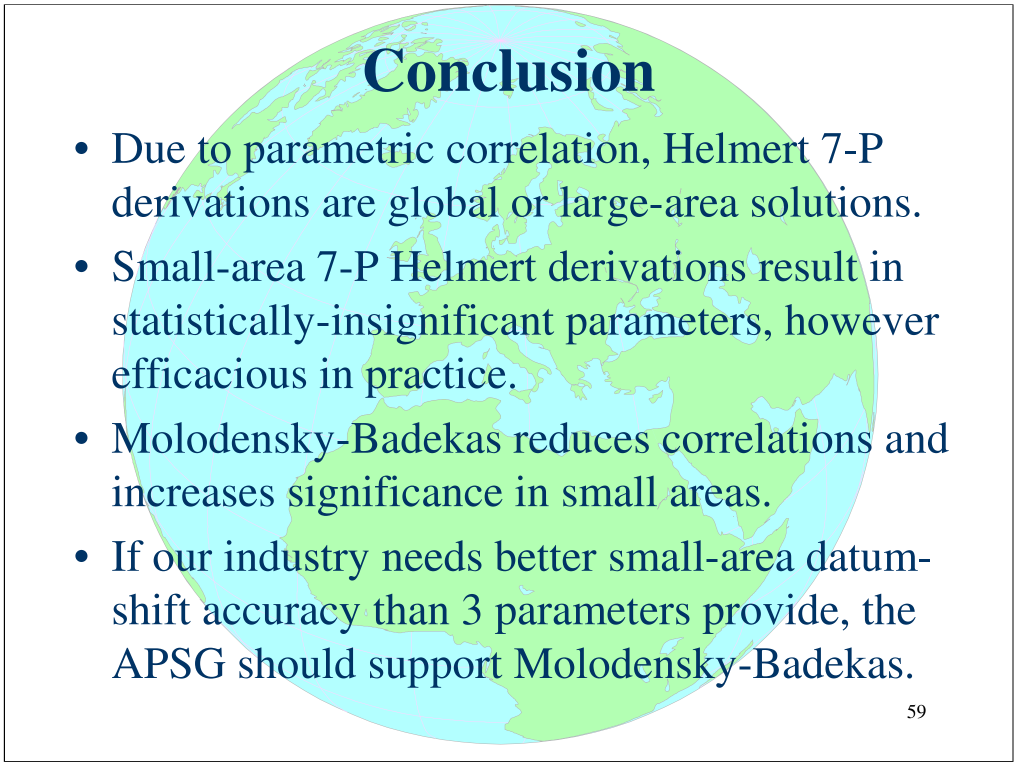
P7DOP							
Area	% of World	Deg (+/-)	Number of Points				
			20	40	80	160	320
World	100	180	0.7	0.5	0.3	0.23	0.16
Hemisphere	50	90	1.0	0.7	0.5	0.3	0.2
Russia	3.35	21.1	3.1	2.1	1.5	1.0	0.7
Australia	1.51	14.1	4.5	3.1	2.2	1.5	1.1
India	0.64	9.2	7.0	4.8	3.3	2.3	1.6
Nigeria	0.18	4.9	13.1	9.0	6.2	4.4	3.1
Germany	0.07	3	21	14.6	10.2	7.1	5.0
Ger No Sea	< 0.01	1.2	53	37	25.4	17.8	12.6
Cyprus	0.002	0.5	128	88	61	42.8	30.2

58

•This slide takes a bit of explaining. The numbers in color are P7DOPs for Helmert 7-p derivations. I like the ones in green, I don't like the ones in red, and I'm not so sure about the ones in between in blue. Those will depend on survey-point quality. But these are my judgements. You need to make up your own minds.

•The "Area" column gives some representative countries of the world. The "% of World" column gives the percentage of the entire surface area of the world that that country represents. Next, I conceptualized that area as a spherical "cap" centered on the X axis in the Gulf of Guinea. Then I computed the angle at the geocenter that, when rotated completely around the X axis, would circumscribe the "cap". By way of further explanation, if you double that angle you would have the range of latitudes of the cap, the "diameter" of the cap. Then, I populated the cap with 20, 40, 80, 160 and 320 uniformly-distributed points and computed the covariance matrix. From the covariance matrix I computed P7DOP using a reference SD of 1. Since, with small areas and fewer points, the P7DOP value fluctuates with different uniform distributions of points, I did this experiment more than 1,000 times in those cases and averaged. When the area is larger and the number of points is greater, P7DOP is much more stable.

•The smaller the P7DOP, the better conditioned is the geometry of the Helmert 7-p derivation. The red numbers demand either Molodensky-Badekas or perhaps even a 3-p shift, certainly not Helmert.



•(Read first three bullets.)

•During the ESRI Petroleum User’s Group meeting in Houston this week we heard that ESRI - in response to user demand - will support Molodensky-Badekas datum transformations in version 9.0 of their GIS product to be released this year.

•Between the extremes of exceedingly-small areas (like Cyprus), where 3-p translations provide adequate “accuracy” (and which may be the only statistically-defendable shift), and large-country or continental datums, where 7-p Helmert shifts are appropriate (but whose definitions we don’t really influence), our industry works in many intermediate areas wherein a M-B 7-p derivation may be the best solution. I’ve given Angola’s Camacupa Datum as an example of a datum where multiple 3-p shifts might be replaced with one M-B 7-p shift.

•I encourage the APSG to go on record in support of industry software developers to provide M-B transformations in our acquisition and data-management applications.

Appendices

- Reverse Molodensky-Badekas Equations
- Reverse M-B Numerical Assessment 1
- Reverse M-B Numerical Assessment 2
- Geographicals to Cartesians
- Cartesians to Geographicals
- References

60

Forward Molodensky-Badekas

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_i = \begin{bmatrix} U \\ V \\ W \end{bmatrix} + \begin{bmatrix} \Delta U \\ \Delta V \\ \Delta W \end{bmatrix} + \begin{bmatrix} 0 & \omega & -\psi \\ -\omega & 0 & \varepsilon \\ \psi & -\varepsilon & 0 \end{bmatrix} \begin{bmatrix} U-U' \\ V-V' \\ W-W' \end{bmatrix}_i + \Delta S \begin{bmatrix} U-U' \\ V-V' \\ W-W' \end{bmatrix}_i$$

$$x = u + \Delta u + R \cdot (u - u') + \Delta S \cdot (u - u')$$

Reverse Molodensky-Badekas

$$u_r = x - \Delta u - R \cdot (x - u') - \Delta S \cdot (x - u')$$

$$\boxed{u_r = u + [R^2 \cdot (u' - u) - R \cdot (2 \cdot \Delta S \cdot (u - u') + \Delta u) + \Delta S^2 \cdot (u' - u) - \Delta S \cdot \Delta u]}$$

61

•The matrix equation for the Molodensky Model for the NGA web site is presented again. Now it is the M-B since the initial point is the barycenter, not the geocenter.

•The second equation on the top is the same, just in a more concise vector notation to save space. The [X, Y, Z] vector is “x”. The rotation matrix is “R”, and so on.

•The third equation, also in vector notation just above the equation in the red box, is the reverse M-B. The LHS (left hand side), the “u_sub_r”, is intended to be the same “u” as the “u” in the second equation. Notice that the third equation replaces the “u” of the RHS of the second equation with the “x” of the LHS of the second equation, which brings into the third equation the RHS of the second equation. Notice also that the signs of the “delta_u”, “R” and “delta_s” are reversed.

•For the Helmert formulation, “u_sub_r” equals “u”. The Helmert is reversible.

•For the M-B formulation, “u_sub_r” equals “u” plus all the other terms in brackets in the fourth equation in the red box.

•That’s the bad news. The good news is that most of the terms in the brackets are very small, products of rotations in radians and scale changes in ppm that are 10^-6 squared. An EPSG Guidance Note makes this point very clearly.

Round-Trip Error: Numerical Assessment 1

Fictitious "Worst Case"

ΔXm	**+700m**
ΔYm	**-500m**
ΔZm	**+200m**
RX"	**-3"**
RY"	**+5"**
RZ"	**-2"**
ΔSppm	**+3ppm**

```
Barycenter
X   =   3655727.05m
Y   =    373465.14m
Z   =   5194453.82m
Lat =     54.905007d
Lon =      5.833039d
Hgt = -1176.849142m
```

Dist km	X dif m	Y dif m	Z dif m
146.56 ,	0.0020 ,	-0.0054 ,	-0.0090
79.22 ,	0.0020 ,	-0.0054 ,	-0.0091
63.48 ,	0.0021 ,	-0.0054 ,	-0.0091
54.42 ,	0.0021 ,	-0.0054 ,	-0.0091
182.60 ,	0.0022 ,	-0.0053 ,	-0.0091
193.61 ,	0.0022 ,	-0.0053 ,	-0.0091
141.44 ,	0.0022 ,	-0.0053 ,	-0.0091
57.33 ,	0.0021 ,	-0.0054 ,	-0.0091
66.45 ,	0.0021 ,	-0.0054 ,	-0.0091
155.94 ,	0.0021 ,	-0.0054 ,	-0.0092
180.01 ,	0.0020 ,	-0.0054 ,	-0.0091
180.93 ,	0.0020 ,	-0.0054 ,	-0.0091
144.05 ,	0.0022 ,	-0.0053 ,	-0.0091
102.29 ,	0.0022 ,	-0.0053 ,	-0.0091
15.08 ,	0.0021 ,	-0.0054 ,	-0.0091
75.51 ,	0.0021 ,	-0.0054 ,	-0.0091
46.36 ,	0.0021 ,	-0.0054 ,	-0.0091
87.70 ,	0.0021 ,	-0.0054 ,	-0.0091
126.00 ,	0.0020 ,	-0.0054 ,	-0.0091

62

•To quantify the statement that these higher order terms are small, I present two numerical assessments of M-B round trip error, both using the fictitious "worst case" datum shift.

•Here are the results for 19 points around a barycenter in the German North Sea.

•The round-trip error in the X axis is 2mm, 5mm in the Y axis and 9mm in the Z axis.

•Note the distances from the barycenter in km.

Round-Trip Error: Numerical Assessment 2

Fictitious "Worst Case"

ΔXm	**+700m**
ΔYm	**-500m**
ΔZm	**+200m**
RX"	**-3"**
RY"	**+5"**
RZ"	**-2"**
ΔSppm	**+3ppm**

```
Barycenter
X   =      6229725.66m
Y   =      -384479.16m
Z   =       -57315.84m
Lat =       -0.529766d
Lon =       -3.531637d
Hgt = -136544.205467m
```

Dist km	X dif m	Y dif m	Z dif m
412.33 ,	0.0020 ,	-0.0053 ,	-0.0088
1544.86 ,	0.0028 ,	-0.0051 ,	-0.0093
616.05 ,	0.0023 ,	-0.0054 ,	-0.0094
1532.70 ,	0.0018 ,	-0.0050 ,	-0.0078
1073.02 ,	0.0025 ,	-0.0052 ,	-0.0094
750.19 ,	0.0017 ,	-0.0055 ,	-0.0089
827.55 ,	0.0018 ,	-0.0053 ,	-0.0084
848.43 ,	0.0023 ,	-0.0051 ,	-0.0087
469.44 ,	0.0020 ,	-0.0053 ,	-0.0087
1630.66 ,	0.0022 ,	-0.0059 ,	-0.0104
1100.34 ,	0.0021 ,	-0.0050 ,	-0.0082
1808.66 ,	0.0015 ,	-0.0052 ,	-0.0079
2446.82 ,	0.0013 ,	-0.0057 ,	-0.0088
1125.73 ,	0.0016 ,	-0.0057 ,	-0.0091
154.16 ,	0.0020 ,	-0.0054 ,	-0.0091
1811.03 ,	0.0029 ,	-0.0053 ,	-0.0100
1780.19 ,	0.0029 ,	-0.0053 ,	-0.0102
539.49 ,	0.0022 ,	-0.0054 ,	-0.0094
1683.97 ,	0.0019 ,	-0.0059 ,	-0.0102

63

•The second numerical assessment is for an area the size of Australia centered in the Gulf of Guinea.

•The results are only slightly larger even though the distances from the barycenter are considerably larger.

•Notice that the size of the parameters of the datum shift dominate the round-trip error, not the distance from the barycenter.

•This is a "worst-case" datum shift, larger than likely to be encountered in practice. Consequently, the round-trip error due to M-B mathematical irreversibility will be less in practice.

Geographicals to Cartesians

Given ellipsoid1 semi-major axis (a) and flattening (f), and latitude (ϕ), longitude (λ), and height (h)

$$b = a - a \cdot f \qquad e^2 = (a^2 - b^2)/a^2 \qquad \nu = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}}$$

$$X = (\nu + h)\cos\phi\cos\lambda$$

$$Y = (\nu + h)\cos\phi\sin\lambda$$

$$Z = (\nu(1 - e^2) + h)\sin\phi$$

64

•These formulas for converting from geographicals to Cartesians and to geographicals again are for completeness and reference only.

•I use Hoar's notation on this slide, but the same equations can be found in Bomford.

•"b" is the semi-minor axis of the ellipsoid. "e^2" is the eccentricity squared of the ellipsoid. "nu" is the radius of curvature in the prime vertical.

•Given these intermediate quantities, we can solve for the Cartesian coordinates.

Cartesians to Geographicals

Given ellipsoid2 a and f, and X, Y and Z Cartesians

$$b = a - a \cdot f \qquad e^2 = (a^2 - b^2)/a^2 \qquad e'^2 = (a^2 - b^2)/b^2$$

$$\nu = \frac{a}{(1 - e^2 \sin^2 \phi)^{1/2}} \qquad p = (X^2 + Y^2)^{1/2} \qquad \theta = \tan^{-1}(\frac{Z \cdot a}{p \cdot b})$$

$$\phi = \tan \frac{Z + e'^2 b \sin^3 \theta}{p - e^2 a \cos^3 \theta}$$

$$\lambda = \tan^{-1}(\frac{Y}{X})$$

$$h = (p/\cos\phi) - \upsilon$$

65

•Cartesians to geographicals are only a bit more complex.

•These formulas are valid for any point on the surface of the earth. A modification is required for points in space.

•"e'^2" is eccentricity prime squared. "p" is the distance from the geocenter to the projection of our point in the equatorial plane.

References

- T. O. Seppelin, The DOD World Geodetic System 1972, May 1974
- Dillon, Gebel, Pryor, *Navigation at the Prime Meridian Revisited*, Navigation, Fall 1977
- C. Boucher, *Investigations on Geodetic Applications of Satellite Doppler Observations for Control Networks*, 2nd International Geodetic Symposium, 1979
- G. Bomford, *Geodesy*, Fourth Edition, 1980, 1983
- G. J. Hoar, *Satellite Surveying*, 1982
- B. R. Harvey, *Transformation of 3D Coordinates*, The Australian Surveyor, June 1986
- DMA Technical Reports 8350.2, 8350.2-A and 8350.2-B, The DOD World Geodetic System 1984, Sept. 1987

- ~~Author: noel.zinn@exxonmobil.com, or ndzinn@houston.rr.com~~
- Author: noel.zinn@hydrometronics.com

66

•These are some references you may consult for further information.